2.3 INTERPRETATION OF KEPLER'S LAWS

Law 1. Whenever two bodies of mass m_1 and m_2 are in free space, such that $m_1 >> m_2$, the body with mass m_2 goes around the body of mass m_1 in an elliptical orbit. This happens when the centre of mass m_1 of the body lies at one of the foci of the ellipse. The body of mass m_2 can be called as the **satellite** of body having mass m_1. The motion in an orbit is governed by the eccentricity e such that

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

where a is major axis and b is minor axis of elliptical path.

The velocity of the satellite is variable from point to point in the orbit path and this creates eccentricity. Theoretically, e can take any value between 0 and 1. When $a = b$, the orbit is circular and there is no eccentricity (change of angular velocity is zero).

Law 2. If the satellite travels from X to Y in an arc distance S_1 in one second during its orbit and during the same period travels arc of distance S_2 (X' to Y') then the area covered under OXY is equal to area covered under $OX'Y'$, as shown in Figure 2.7.

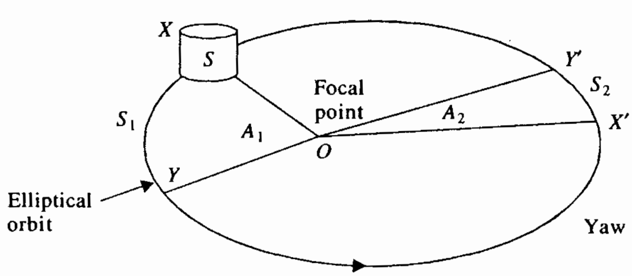

Figure 2.7 Kepler's law.

That means as the satellite moves away from the earth its velocity decreases, while it moves faster when it is nearer to the primary body.

Law 3. If the period of orbit of satellite is t_0 and the mean distance between the primary body and satellite is d_0, then square of the periodic time of orbit is proportional to the cubic distance. Such that

$$t_0^2 = ad_0^3$$

where a is called **orbital constant** or **Kepler's constant**.

Satellites in Circular Orbits

Most of the satellites used for communication purpose move in circular orbit and are preferred to be parked in geostationary orbit. If the satellite is in any other orbit the relative velocity between earth station and the satellites will not be zero and more than one satellite will be required for uninterrupted communication. Think of what would happen if every country depends on satellites not in geostationary orbits? Therefore, our main interest would be to study satellites in circular orbit in the equatorial plane, moving in space, free from aerodynamic lift or any influence of external torques.

Let M be the mass of earth and m be the mass of satellite. If the satellite is moving with an angular velocity ω under the influence of earth's gravitation G and its own gravitation, the satellite would go into circular orbit provided it satisfies the conditions derived hereunder. [Refer Figure 2.8]

For an orbit height of h or orbit radius $(r_e + h)$, the velocity of satellite would be

$$v_s = \omega(r_e + h)$$

where r_e is the radius of earth.

If the satellite has to orbit stably, then the centrifugal force that tries to take away the satellite should be equal to the centripetal force with which the satellite is attracted towards the earth.

If F_1 is the force acting upon by satellite due to its own mass m and velocity v, the satellite tends to pull away from earth, such that

$$F_1 = \frac{mv^2}{d} = \frac{mv_s^2}{r_e + h} = \frac{m\omega^2(r_e + h)^2}{r_e + h}$$

$$F_1 = m(2\pi/T)^2\ (r_e + h) \qquad 2.1$$

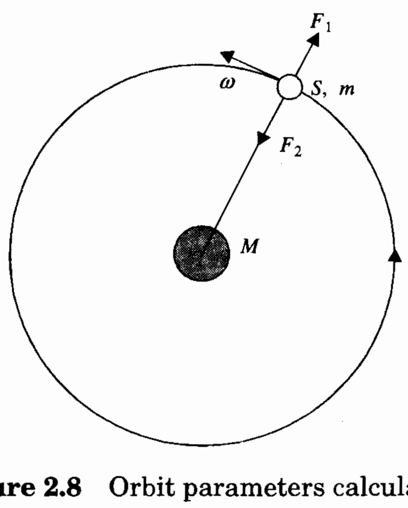

Figure 2.8 Orbit parameters calculation.

On the other hand, F_2, the force that attracts the satellite towards earth is given by

$$F_2 = \frac{GMm}{d^2} = \frac{GMm}{(r_e + h)^2} \qquad 2.2$$

In equilibrium the centrifugal force should be equal to the centripetal force, for example, if a stone tied to a string is made to orbit by rotating the hand holding it, the stone will go round and round till the force acting on it due to motion of hand is equal to the inertial force that is trying to take the stone away. In conclusion,

$$F_1 = F_2$$

$$\therefore \quad \frac{m(2\pi)^2\ (r_e + h)}{T^2} = \frac{GMm}{(r_e + h)^2} \qquad 2.3$$

If any one of the forces increases or decreases, the stone no longer remains in orbit. For example, if the motion of hand disturbs or reduces, the stone will fall on the head or on the other hand if the string breaks F_1 becomes very large and F_2 tends to zero making the stone fly away.

Rearranging Eq. (2.3), the time of orbit would be

$$T_s^2 = \frac{4\pi^2\ (r_e + h)^3}{g_0} \qquad 2.4a$$

$$T_s = \frac{2\pi}{\sqrt{g_0}}(r_e + h)^{3/2} \qquad 2.4b$$

where $g_0 = GM$ called the **gravitational coefficient** or **Kepler's coefficient**.

For earth, $G = 6.672 \times 10^{-11}$ newton metre/kg^2 and $M = 5.97 \times 10^{24}$ kg. Therefore,

$$g_0 = 3.9861 \times 10^5\ \text{km}^3/\text{s}^2$$

Is it not the same as predicted by Kepler in his third law?

i.e.

$$t_0^2 = ad_0^3$$

$$\text{where } a = \frac{2\pi}{\sqrt{g_0}}$$

Equation (2.4b) indicates that, as the radius of orbit increases, the time of the orbit also increases.

Now, let us find the velocity of satellite. Substituting $v_s = \omega(r_e + h)$ $= (2\pi/T)(r_e + h)$, we get

$$v_s = \sqrt{\frac{GM}{(r_e + h)}} = \sqrt{\frac{g_0}{(r_e + h)}} \qquad 2.5$$

This shows with the increase in orbit radius the velocity of the satellite decreases as predicted by Kepler in his second law.

For such a satellite to look stationary in geostationary orbit, it is necessary that the relative velocity between the orbiting satellite and orbiting earth (earth orbits round the sun) should be zero. That means, at sub-satellite point S' the velocity of the satellite and earth should be same, i.e., 7.905364 km/s and as the satellite goes farther away from the earth the velocity should correspondingly decrease. Also if the satellite has to be geosynchronous, the time taken for one full orbit should be 24 hrs. The solar day of 24 hrs does not give exact calculations as both sun and earth are moving objects, therefore, sidereal day referred to fixed stars is used for better accuracy in orbiting satellites. A sidereal day is of 23 hrs 56 mins and 4 seconds which is 0.9972 times the solar day.

Additionally for the satellite to orbit in space, it should also revolve round its pitch axis like a top to create necessary gyroscopic stiffness. Imagine a revolving top, when this top attains a certain angular velocity, it revolves around its pitch axis stably without falling and at higher speeds even leaves the surface. This is possible because of the stiffness it attains due to high angular velocity. But as the velocity decreases the top tends to wobble. This is analogous to a revolving satellite.

Example 2.1

A satellite is orbiting in a geosynchronous orbit of radius 41500 km. Find the velocity and time of orbit. What will be the change in velocity if the radius reduces to 36000 km. If g_0 = 398600.5 km^3/s^2.

Solution

Given:

The gravitational coefficient g_0 = 398600.5 km^3/s^2
Radius of orbit = 41500 km

Since

$$v_s = \sqrt{\frac{g_0}{(r_e + h)}} = \sqrt{\frac{398600.5}{41500}} = 3.099 \text{ km/s}$$

and

$$\text{Period of orbit } T_s = \frac{2\pi d^{3/2}}{\sqrt{g_0}} = \frac{2\pi(41500)^{3/2}}{\sqrt{398600.5}} = 84136.26 \text{ s}$$

For $(r_e + h)$ = 36000 km

$$v_s = \sqrt{\frac{g_0}{(r_e + h)}} = \sqrt{\frac{398600.5}{36000}} = 3.3274 \text{ km/s}$$

Thus,

Increase in velocity = 3.3274 – 3.099 = 0.2284 km/s

Ans.

Some important parameters of circular orbit geosynchronous satellite are given in Table 2.1

Table 2.1 Parameters of Circular Orbit Geosynchronous Satellite

Quantity	*Equation*	*At sub-satellite point*	*At geostationary orbit*	*Units*
Velocity	$v_s = \sqrt{\frac{g_0}{(r_e + h)}}$	7.905364	3.074689	km/s
Orbit period	$T_s = \frac{2\pi d^{3/2}}{\sqrt{g_0}}$	5069.347	86164.091	s
Angular velocity	$\omega_s = \frac{\sqrt{g_0}}{d^{3/2}}$	0.001239	72.9211×10^{-6}	rad/s
Acceleration	$a = \frac{g_0}{d^2}$	9.79×10^{-3}	0.22×10^{-3}	km/s^2

Practical Problems in Space and Orbit Perturbations

As a first approximation, we have assumed that only mutual gravitational forces of the two bodies in space (earth and satellite) act upon the orbiting satellite in space. The orbit described so far is Keplerian orbit. However, the Keplerian orbit is ideal in the sense that it assumes that the earth is a uniform spherical mass and that the only force acting is the centrifugal force, resulting from satellite motion balancing the gravitational pull of the earth.

In actual practice it is not true because any satellite in space is also acted upon by torque due to other external as well as internal forces.

External torques

1. Gravitational effect of other planetary bodies like sun, moon and other heavenly bodies.
2. Non-spherical earth.
3. Atmospheric drag (only in low orbits).
4. Aerodynamic forces (only in low orbits).
5. Solar pressure on the solar cell panel.
6. Magnetic forces acting on the satellite due to earth's magnetic fields.

As is seen above, the gravitational force on the satellite is given by

$$F = \frac{g_0 m}{d^2} \qquad 2.6$$